Exam. Code : 211004
Subject Code : 4634

M.Sc. (Mathematics) 4th Semester

MATH-581 : FUNCTIONAL ANALYSIS—II

Time Allowed—2 Hours] [Maximum Marks—100

Note :— Attempt any ***four*** questions. All questions carry equal marks.

1. (a) Define weak convergence and strong convergence of a sequence $\{x_n\}$ in a normed linear space X. If $\{x_n\}$ is weakly convergent to $x_0 \in X$, show that $x_0 \in \overline{Y}$, where $Y = \text{span}\{x_n\}$. 10

   (b) Prove that in a finite dimensional normed linear space, a sequence is weakly convergent if and only if it is strongly convergent. 10

2. (a) Define self-adjoint operator on a Hilbert space. If H is a Hilbert space over $\mathbb{C}$ and T is a bounded linear operator on H, prove that T is self-adjoint if and only if $<T(x), x>$ is real for all x. 10

   (b) Let H be a Hilbert space cover $\mathbb{C}$ and T be a bounded linear operator on H. Prove that T preserves inner products on H if and only if T preserves norms. Also, prove that any unitary operator on a Hilbert space preserves norms but a norm preserving bounded linear operator on a Hilbert space may not be unitary. 10

3. (a) Let H be a Hilbert space over $\mathbb{C}$. If P is a projection on H with range M and null space N, prove that $M \perp N$ if and only if P is self-adjoint. Also prove that in this case $N = M^{\perp}$. 10

(b) Let T be a normal operator on a Hilbert space H. If k is a spectral value of T, prove that there exists a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ for each n such that $T(x_n) - kx_n \to 0$. 10

4. State and prove Spectral theorem for normal operators on a finite dimensional Hilbert space. 20

5. (a) Prove that every compact linear map between two normed linear spaces is continuous but a continuous linear map may not be compact. Prove however that a continuous linear map of finite rank is compact and conversely if F is a compact linear map between two Banach spaces X and Y such that range F is closed in Y, then F is continuous and of finite rank. 10

(b) Let X and Y be normed spaces and $F : X \to Y$ be a compact linear map. If $\{x_n\}$ is a sequence in X converging weakly to x in X, prove that $\{F(x_n)\}$ converges to F(x) in Y. 10

6. (a) If T is a compact linear operator on a normed linear space X, prove that every eigenspace of T corresponding to a nonzero eigenvalue of T is finite dimensional. 10

(b) Let X be a normed space and T be a compact linear operator on X. Prove that the eigen spectrum and the spectrum of T are countable sets and have 0 as the only possible limit point. 10

7. (a) Define a Banach algebra. Prove that the set of all regular elements of a complex Banach algebra A is an open subset of A. 10

(b) Let A be a complex Banach algebra, S and Z denote the set of singular elements of A and set of topological divisors of zero in A respectively. Prove that Z is a subset of S and boundary of S is a subset of Z. 10

8. (a) Define spectrum of an element of a complex Banach algebra. Prove that spectrum of any element of a complex Banach algebra is non-empty. 10

(b) Prove that spectral radius of any element x of a complex Banach algebra A equals lim $\|x^n\|^{1/n}$. 10